

ON THE PLACEMENT OF AN OBSTACLE SO AS TO OPTIMIZE THE DIRICHLET HEAT TRACE

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ABSTRACT. We prove that among all doubly connected domains of \mathbb{R}^n bounded by two spheres of given radii, $Z(t)$, the trace of the heat kernel with Dirichlet boundary conditions, achieves its minimum when the spheres are concentric (i.e., for the spherical shell). The supremum is attained when the interior sphere is in contact with the outer sphere.

This is shown to be a special case of a more general theorem characterizing the optimal placement of a spherical obstacle inside a convex domain so as to maximize or minimize the trace of the Dirichlet heat kernel. In this case the minimizing position of the center of the obstacle belongs to the “heart” of the domain, while the maximizing situation occurs either in the interior of the heart or at a point where the obstacle is in contact with the outer boundary.

Similar statements hold for the optimal positions of the obstacle for any spectral property that can be obtained as a positivity-preserving or positivity-reversing transform of $Z(t)$, including the spectral zeta function and, through it, the regularized determinant.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 Euclidean domain and let

$$\lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \cdots \leq \lambda_i(\Omega) \leq \cdots \rightarrow \infty,$$

be the sequence of eigenvalues of the Dirichlet realization of the Laplacian $-\Delta$ in Ω , where each eigenvalue is repeated according to its multiplicity. The corresponding “heat operator” $e^{t\Delta}$ has finite trace for all

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$t > 0$ (known in physical literature as the “partition function”), which we denote

$$Z_{\Omega}(t) = \sum_{k \geq 1} e^{-\lambda_k(\Omega)t}. \quad (1)$$

Let ζ_{Ω} be the zeta function, defined as the meromorphic extension to the entire complex plane of $\sum_{k=1}^{\infty} \lambda_k(\Omega)^{-s}$, which is known to be convergent and holomorphic on $\{\operatorname{Re} s > \frac{n}{2}\}$. Following [29], we denote by $\det(\Omega)$ the regularized determinant of the Dirichlet Laplacian in Ω defined by

$$\det(\Omega) = \exp(-\zeta'_{\Omega}(0)). \quad (2)$$

Eigenvalue optimization problems date from Rayleigh’s “Theory of Sound” (1877), where it was suggested that the disk should minimize the first eigenvalue λ_1 among all planar domains of given measure. Rayleigh’s conjecture was proved in any dimension independently by Faber [10] and Krahn [22].

Later, Luttinger [25] proved an isoperimetric result analogous to Faber-Krahn for $Z(t)$, considered as a functional on the set of bounded Euclidean domains, that is, for any bounded domain $\Omega \subset \mathbb{R}^n$ and any $t > 0$, Luttinger showed that

$$Z_{\Omega}(t) \leq Z_{\Omega^*}(t),$$

where Ω^* is a Euclidean ball whose volume is equal to that of Ω . A similar property was proved in [26] for the regularized determinant of the Laplacian in two dimensions (see [23, 30] for other examples of results in this direction).

The case of multiply connected planar domains, i.e. whose boundary admits more than one component, was first considered by Hersch. Using the method of interior parallels, in [19] Hersch proved the following extremal property of annular membranes:

“A doubly connected fixed membrane, bounded by two circles of given radii, has maximum λ_1 when the circles are concentric”.

Hersch’s result has been extended to a wider class of domains in any dimension by Harrell, Kröger and Kurata [15] and Kesavan [21], whereby the authors consider a fixed domain D from which an “obstacle” of fixed shape, usually spherical, has been excised. The position of the obstacle is allowed to vary, and the problem is to maximize or minimize λ_1 . The critical assumption on the domain D in [15] is an “interior symmetry property,” and with this assumption the authors further proved that, for the special case of two balls, λ_1 decreases when the center of the small ball (the obstacle) moves away from the center of the large ball, using a technique of domain reflection. For a wider class

of domains containing obstacles, it was shown in [15] that the maximizing position of the obstacle resides in a special subset of D , which in the case where D is convex corresponds to what has later come to be called the heart of D in [2, 3], denoted $\heartsuit(D)$ (see the definition below). El Soufi and Kiwan [7, 8] have moreover proved other extensions of Hersch's result including one valid for the second eigenvalue λ_2 .

The main aim of this paper is to establish a Hersch-type extremal result for the heat trace, the spectral zeta function, and the determinant of the Laplacian, as well as suitable generalizations for more general outer domains. We begin by stating the special case of domains bounded by balls: Given two positive numbers $R > r$ and a point $\mathbf{x} \in \mathbb{R}^n$, $|\mathbf{x}| < R - r$, we denote by $\Omega(\mathbf{x})$ the domain of \mathbb{R}^n obtained by removing the ball $B(\mathbf{x}, r)$ of radius r centered at \mathbf{x} from within the ball of radius R centered at the origin.

Theorem 1.1. *Let $\Omega(\mathbf{x})$ be the domain bounded by balls as in the preceding paragraph.*

(i) *For every $t > 0$, the heat trace $Z_{\Omega(\mathbf{x})}(t)$ is nondecreasing as the point \mathbf{x} moves from the origin directly towards the boundary of the larger ball. In particular, $Z_{\Omega(\mathbf{x})}(t)$ is minimal when the balls are concentric ($\mathbf{x} = O$) and maximal when the small ball is in contact with the boundary of the larger ball ($|\mathbf{x}| = R - r$).*

(ii) *For every $s > 0$, the zeta function $\zeta_{\Omega(\mathbf{x})}(s)$ increases as the point \mathbf{x} moves from the origin directly towards the boundary of the larger ball. In particular, $\zeta_{\Omega(\mathbf{x})}$ is minimal when the balls are concentric and maximal in the limiting situation when the small ball approaches the boundary of the larger ball.*

(iii) *The determinant of the Laplacian $\det(\Omega(\mathbf{x}))$ decreases as the point \mathbf{x} moves from the origin directly towards the boundary of the larger ball. In particular, $\det(\Omega(\mathbf{x}))$ is maximal when the balls are concentric and minimal in the limiting situation when the small ball approaches the boundary of the larger ball.*

Let us clarify what we mean by the “limiting situation when the ball approaches the boundary” in this and in the more general Theorem 1.2. Our conclusion in these cases is derived by contradiction. By assuming that the obstacle is in the interior of the domain, extremality can be excluded. If the function under consideration is continuous, then, since the set of configurations including the cases where the obstacle touches the boundary is compact, the extremum is attained there. Since the heat trace is continuous with respect to translations of the obstacle, claim (i) is not problematic, and the same is true for claim (ii) when $s >$

$\frac{n}{2}$. For smaller real values of s the spectral zeta function is defined by analytic continuation, and the determinant of the Laplacian is defined by (2). In the proof these quantities will be shown to be continuous as a function of the position of the obstacle as it moves to the boundary, and their limits are what we describe as the “limiting situations” of the theorems. The subtlety here is that when the obstacle is in contact with the boundary, a cusp is formed, as a consequence of which the heat-trace asymptotics are not necessarily known well enough to allow a direct definition of $\zeta(s)$, $s \leq \frac{n}{2}$, by analytic continuation, cf. [31].

Since $e^{-\lambda_1(\Omega)t}$ is the leading term in $Z_\Omega(t)$ as t goes to infinity, it is clear that Theorem 1.1 implies the optimization result mentioned above for λ_1 .

In order to state the more general theorem of which Theorem 1.1 is a special case, we recall some definitions.

Definition 1.1. *Let P be a hyperplane in \mathbb{R}^n which intersects D so that $D \setminus P$ is the union of two open subsets located on either side of P . According to [15], the domain D is said to have the interior reflection property with respect to P if the reflection through P of one of these subsets, denoted D_s , is contained in D . Any such P will be called a hyperplane of interior reflection for D . The subdomain D_s will be called the small side of D (with respect to P) and **its complement** $D_b = D \setminus D_s$ will be called the big side.*

When D is convex, the heart of D is defined as the intersection of all the big sides with respect to the hyperplanes of interior reflection of D . We denote it $\heartsuit(D)$.

This definition of $\heartsuit(D)$ is equivalent to that introduced in [2, 3], where several properties of the heart of a convex domain are investigated. A point \mathbf{x} belongs to $\heartsuit(D)$ if either there is no hyperplane of interior reflection passing through \mathbf{x} or if any hyperplane of interior reflection passing through \mathbf{x} is such that the reflection of $\partial D_s \setminus P$ touches ∂D_b . The first situation occurs when \mathbf{x} is an interior point of $\heartsuit(D)$ while the latter is characteristic of the boundary points of $\heartsuit(D)$.

By construction the heart of a bounded convex domain D is a nonempty relatively closed subset of D . Moreover, if D is strictly convex and bounded, then $\text{dist}(\heartsuit(D), \partial D) > 0$. We observe that for the ball and for many other domains with sufficient symmetry to identify an unambiguous center point, the heart is simply the center. It is, however, shown in [2] that without reflection symmetries the typical heart has non-empty interior, even for simple polygons. For instance, for an asymmetric acute triangle, it is a quadrilateral bounded by two angle

bisectors and two perpendicular axes, while for an asymmetric obtuse triangle, it can be either a quadrilateral or a pentagon.

Theorem 1.2. *Let D be a bounded C^2 convex domain of \mathbb{R}^n and let $r > 0$ be such that $D_r = \{\mathbf{x} \in D : \text{dist}(\mathbf{x}, \partial D) > r\} \neq \emptyset$. For every $\mathbf{x} \in \bar{D}_r$ we set $\Omega(\mathbf{x}) = D \setminus B(\mathbf{x}, r)$.*

(i) *For each fixed $t > 0$, the function $\mathbf{x} \in \bar{D}_r \mapsto Z_{\Omega(\mathbf{x})}(t)$ achieves its minimum at a point $\mathbf{x}_0(t) \in \heartsuit(D)$, while the maximum is achieved either at an interior point $\mathbf{x}_1(t)$ of $\heartsuit(D)$ or in the limiting situation when the ball approaches the boundary of D .*

(ii) *For each fixed $s > 0$ and $\mathbf{x} \in D_r \setminus \heartsuit(D)$, the zeta function satisfies*

$$\zeta_{\Omega(\mathbf{x})}(s) > \inf_{\mathbf{y} \in \heartsuit(D) \cap D_r} \zeta_{\Omega(\mathbf{y})}(s), \quad (3)$$

and $\zeta_{\Omega(\mathbf{x})}(s)$ is less than the supremum of all the values attained by the zeta function in the limiting situations when the ball approaches the boundary of D . Moreover, if $r < \text{dist}(\heartsuit(D), \partial D)$, then $\mathbf{x} \in D_r \mapsto \zeta_{\Omega(\mathbf{x})}(s)$ achieves its infimum at a point $\mathbf{x}_0(s) \in \heartsuit(D)$, while the supremum is reached either at an interior point $\mathbf{x}_1(s)$ of $\heartsuit(D)$ or in a limiting situation when the ball approaches the boundary of D .

(iii) *The regularized determinant of the Laplacian satisfies*

$$\det(\Omega(\mathbf{x})) < \sup_{\mathbf{y} \in \heartsuit(D) \cap D_r} \det(\Omega(\mathbf{y})) \quad (4)$$

for every $\mathbf{x} \in D_r \setminus \heartsuit(D)$, and $\det(\Omega(\mathbf{x}))$ is greater than the infimum of all the values attained by the determinant in the limiting situations when the ball approaches the boundary of D . Moreover, if $r < \text{dist}(\heartsuit(D), \partial D)$, then the function $\mathbf{x} \in D_r \mapsto \det(\Omega(\mathbf{x}))$ achieves its supremum at a point $\mathbf{x}'_0 \in \heartsuit(D)$, while the infimum is achieved either at an interior point \mathbf{x}'_1 of $\heartsuit(D)$ or in a limiting situation when the ball approaches the boundary of D .

We remark that a straightforward consideration of the limit $t \rightarrow \infty$ leads back to related results of [15]. As in that article, it is not difficult to extend Theorem 1.2 to many nonconvex domains, at the price of entering into the sometimes complex nature of $\heartsuit(D)$. For simplicity we limit the present article to the case of convex D .

We conjecture that the minimum of $Z_{\Omega(\mathbf{x})}(t)$ is never achieved outside $\heartsuit(D)$, and that the maximum of $Z_{\Omega(\mathbf{x})}(t)$ and resp. the maximum of $\zeta_{\Omega(\mathbf{x})}(s)$ and the minimum of $\det(\Omega(\mathbf{x}))$, are achieved only in the limiting situation where $B(\mathbf{x}, r)$ touches the boundary of D . This is certainly the case for example when a convex domain D admits a hyperplane of symmetry since then, $\text{int}(\heartsuit(D)) = \emptyset$.

Remark 1.1. *We shall approach the analysis of the spectral zeta function and the regularized determinant through order-preserving integral transforms relating them to the heat trace. As in [14], transform theory can similarly be used to obtain corollaries for many further functions, e.g., Riesz means, with respect to the optimal position of an obstacle.*

A main ingredient of the proof is the following Hadamard-type formula for the first variation of $Z_\Omega(t)$ with respect to a deformation $\Omega_\varepsilon = f_\varepsilon(\Omega)$ of the domain :

$$\frac{\partial}{\partial \varepsilon} Z_{\Omega_\varepsilon}(t) \Big|_{\varepsilon=0} = -\frac{t}{2} \int_{\partial \Omega} \Delta K(t, \mathbf{x}, \mathbf{x}) v(\mathbf{x}) dx, \quad (5)$$

where $v = X \cdot \nu$ is the component of the deformation vectorfield $X = \frac{df_\varepsilon}{d\varepsilon} \Big|_{\varepsilon=0}$ in the direction of the inward unit normal ν , and K is the heat kernel (cf. [6, Theorem 4.1]). Notice that this formula coincides with that given by Ozawa in [27, Theorem 4] for deformations of the form $f_\varepsilon(\mathbf{x}) = \mathbf{x} + \varepsilon \rho(\mathbf{x}) \nu(\mathbf{x})$ along the boundary, where ρ is a smooth function on $\partial \Omega$. Indeed, it is easy to check that for all $\mathbf{x} \in \partial \Omega$, $v(\mathbf{x}) = \rho(\mathbf{x})$ and (using (12) below) $\Delta K(t, \mathbf{x}, \mathbf{x}) = 2 \sum_{k=0}^{\infty} e^{-\lambda_k t} |\nabla u_k(\mathbf{x})|^2 = 2 \sum_{k=0}^{\infty} e^{-\lambda_k t} \left| \frac{\partial u_k}{\partial \nu}(\mathbf{x}) \right|^2$. For more information about Hadamard deformations we refer to [5, 6, 11, 12, 18, 27, 29].)

2. PROOF OF RESULTS

Let $\Omega \subset \mathbb{R}^n$ be a domain of the form $\Omega = D \setminus B$, where D is a bounded domain and B is a convex domain such that the closure of B is contained in D . (For simplicity our theorems have been stated for the case of a spherical obstacle B , but the essential argument requires only a lower degree of symmetry.)

Let us start by establishing how the zeta function and the Laplacian determinant are related to the heat trace in our situation. Indeed, the following formula is valid for every complex number s with $\text{Re } s > \frac{n}{2}$:

$$\zeta_\Omega(s) := \sum_{k \geq 1} \frac{1}{\lambda_k^s(\Omega)} = \frac{1}{\Gamma(s)} \int_0^\infty Z_\Omega(t) t^{s-1} dt.$$

It is well known that the function Z_Ω satisfies

$$Z_\Omega(t) \sim \sum_{k \geq 0} a_k t^{\frac{(k-n)}{2}} \quad \text{as } t \rightarrow 0, \quad (6)$$

where a_k is a sequence of real numbers that depend only on the geometry of the boundary of Ω (see e.g. [1]). In particular, **the coefficients**

a_k are independent of the position of B within D . We set

$$\tilde{Z}_\Omega(t) = Z_\Omega(t) - \sum_{k=0}^n a_k t^{\frac{(k-n)}{2}},$$

so that $\tilde{Z}_\Omega(t)/\sqrt{t}$ is a bounded function in a neighborhood of $t = 0$. We also introduce the meromorphic function

$$R(s) = \frac{1}{\Gamma(s)} \sum_{k=0}^n a_k \int_0^1 t^{s-1+(k-n)/2} dt = \frac{1}{\Gamma(s)} \sum_{k=0}^n \frac{a_k}{s - (n-k)/2},$$

which has poles at $1/2, 1, 3/2, 2, \dots, n/2$. (Note that $s = 0$ is not a pole since $\frac{1}{s\Gamma(s)}$ is a holomorphic function on \mathbb{C} .) Consequently, for every $s \in \mathbb{R}^+$,

$$\zeta_\Omega(s) = R(s) + \frac{1}{\Gamma(s)} \int_0^1 \tilde{Z}_\Omega(t) t^{s-1} dt + \frac{1}{\Gamma(s)} \int_1^\infty Z_\Omega(t) t^{s-1} dt. \quad (7)$$

where the last term is an entire function of s , since $Z_\Omega(t)$ behaves as $e^{-\lambda_1(\Omega)t}$ when $t \rightarrow +\infty$. On the other hand, the reciprocal gamma function $f(s) := \frac{1}{\Gamma(s)}$ vanishes at $s = 0$ and satisfies $f'(0) = 1$. Therefore,

$$\zeta'_\Omega(0) = R'(0) + \int_0^1 \tilde{Z}_\Omega(t) t^{-1} dt + \int_1^\infty Z_\Omega(t) t^{-1} dt. \quad (8)$$

Assume that the domain D has the interior reflection property with respect to a hyperplane P about which the set B is reflection-symmetric. (Here we do not need to restrict to convex D .) Our strategy is to consider a displacement of the obstacle by ε in a certain direction and to show that $Z_{\Omega_\varepsilon}(t)$ is monotonically increasing in that direction.

Thus let V be the unit vector perpendicular to P and pointing in the direction of the small side D_s . For small $\varepsilon > 0$, we translate B by a distance ε in the direction of V and set $B_\varepsilon := B + \varepsilon V$ and $\Omega_\varepsilon := D \setminus B_\varepsilon$. The results of this paper rely on the following proposition.

Proposition 2.1. *Assume that the domain D has the interior reflection property with respect to a hyperplane P about which the set B is reflection-symmetric. Consider displacements as described above. Then,*

$$\frac{\partial}{\partial \varepsilon} Z_{\Omega_\varepsilon}(t) \Big|_{\varepsilon=0} > 0, \quad (9)$$

except possibly for a finite set of values t in any interval $[\tau, \infty)$ with $\tau > 0$. Moreover, for each $s > 0$,

$$\frac{d}{d\varepsilon} \zeta_{\Omega_\varepsilon}(s) \Big|_{\varepsilon=0} > 0, \quad (10)$$

and

$$\frac{d}{d\varepsilon} \det(\Omega_\varepsilon) \Big|_{\varepsilon=0} < 0. \quad (11)$$

Proof. The heat kernel K on Ω under the Dirichlet boundary condition is defined as the fundamental solution of the heat equation, that is

$$\begin{cases} (\frac{\partial}{\partial t} - \Delta_y)K(t, \mathbf{x}, \mathbf{y}) = 0 & \text{in } \Omega \\ K(0^+, \mathbf{x}, \mathbf{y}) = \delta_{\mathbf{x}}(\mathbf{y}) \\ K(t, \mathbf{x}, \mathbf{y}) = 0 & \forall \mathbf{y} \in \partial\Omega. \end{cases}$$

The relationship between the heat kernel and the spectral decomposition of the Dirichlet Laplacian in Ω is given by

$$K(t, \mathbf{x}, \mathbf{y}) = \sum_{k \geq 1} e^{-\lambda_k(\Omega)t} u_k(\mathbf{x}) u_k(\mathbf{y}), \quad (12)$$

where $(u_k)_{k \geq 1}$ is an $L^2(\Omega)$ -orthonormal family of eigenfunctions satisfying

$$\begin{cases} -\Delta u_k = \lambda_k(\Omega) u_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

The heat trace is then given by

$$Z_\Omega(t) = \int_\Omega K(t, \mathbf{x}, \mathbf{x}) dx = \sum_{k \geq 1} e^{-\lambda_k(\Omega)t}.$$

Let X be a smooth vectorfield such that X vanishes on ∂D and coincides with the vector V on ∂B . For sufficiently small ε , one has $\Omega_\varepsilon = f_\varepsilon(\Omega)$ where $f_\varepsilon(\mathbf{x}) = \mathbf{x} + \varepsilon X(\mathbf{x})$. The Hadamard-type formula (5) gives :

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} Z_{\Omega_\varepsilon}(t) \Big|_{\varepsilon=0} &= -\frac{t}{2} \int_{\partial\Omega} \Delta K(t, \mathbf{x}, \mathbf{x}) (X \cdot \nu)(\mathbf{x}) dx \\ &= -\frac{t}{2} \int_{\partial B} \Delta K(t, \mathbf{x}, \mathbf{x}) V \cdot \nu(\mathbf{x}) dx. \end{aligned}$$

Let B_s be the half of B contained in the small side D_s of D and $(\partial B)_s = \partial B \cap D_s$. Here, we assume without loss of generality that D_s is connected (otherwise, B_s is contained in one connected component of D_s and we concentrate our analysis on this single component).

Using the symmetry assumption on B with respect to P we obtain

$$\frac{\partial}{\partial \varepsilon} Z_{\Omega_\varepsilon}(t) \Big|_{\varepsilon=0} = -\frac{t}{2} \int_{(\partial B)_s} (\Delta K(t, \mathbf{x}, \mathbf{x}) - \Delta K(t, \mathbf{x}^*, \mathbf{x}^*)) V \cdot \nu(\mathbf{x}) dx \quad (13)$$

where \mathbf{x}^* stands for the reflection of \mathbf{x} through P .

Define the function $\phi(t, \mathbf{x}, \mathbf{y}) = K(t, \mathbf{x}, \mathbf{y}) - K(t, \mathbf{x}^*, \mathbf{y}^*)$ on $(0, \infty) \times \Omega_s \times \Omega_s$ with $\Omega_s = D_s \setminus B_s$.

Claim : For all $(t, \mathbf{x}, \mathbf{y}) \in (0, \infty) \times \Omega_s \times \Omega_s$, $\phi(t, \mathbf{x}, \mathbf{y}) \leq 0$.

Let us check the sign of $\phi(t, \mathbf{x}, \mathbf{y})$ on $(0, \infty) \times \partial\Omega_s \times \partial\Omega_s$. Notice that $\partial\Omega_s$ is the union of three components : $(\partial D)_s$, $(\partial B)_s$ and $\Omega \cap P$. First, from the boundary conditions, if $\mathbf{x} \in (\partial B)_s$ or $\mathbf{y} \in (\partial B)_s$, then $K(t, \mathbf{x}, \mathbf{y}) = K(t, \mathbf{x}^*, \mathbf{y}^*) = 0$ and, hence, $\phi(t, \mathbf{x}, \mathbf{y}) = 0$. On the other hand, $K(t, \mathbf{x}, \mathbf{y})$ vanishes as soon as $\mathbf{x} \in (\partial D)_s$ or $\mathbf{y} \in (\partial D)_s$, which implies $\phi(t, \mathbf{x}, \mathbf{y}) = -K(t, \mathbf{x}^*, \mathbf{y}^*) \leq 0$. It remains to consider the case where both \mathbf{x} and \mathbf{y} belong to $\Omega \cap P$. In this case we have $\mathbf{x}^* = \mathbf{x}$, $\mathbf{y}^* = \mathbf{y}$ and $\phi(t, \mathbf{x}, \mathbf{y}) = 0$.

Observe next that for all $\mathbf{x} \in \bar{\Omega}_s$, the function $(t, \mathbf{y}) \mapsto \phi(t, \mathbf{x}, \mathbf{y})$ is a solution of the following parabolic problem :

$$(*) \begin{cases} (\frac{\partial}{\partial t} - \Delta_y) \phi(t, \mathbf{x}, \mathbf{y}) = 0 & \text{in } \Omega_s \\ \phi(0^+, \mathbf{x}, \mathbf{y}) = 0. \end{cases}$$

Given any $\mathbf{x} \in \partial\Omega_s$, the parabolic maximum principle (see e.g., [9], §7.1) tells us that, since $(t, \mathbf{y}) \mapsto \phi(t, \mathbf{x}, \mathbf{y})$ is nonpositive on the boundary of the cylinder $(0, \infty) \times \Omega_s$, it follows that $\phi(t, \mathbf{x}, \mathbf{y}) \leq 0$ for all $t > 0$ and all $\mathbf{y} \in \bar{\Omega}_s$.

Now, from the symmetry of ϕ with respect to \mathbf{x} and \mathbf{y} , the function $(t, \mathbf{x}) \mapsto \phi(t, \mathbf{x}, \mathbf{y})$ satisfies the same parabolic system as $(*)$. Since we have established that $\forall \mathbf{y} \in \bar{\Omega}_s$, the function $(t, \mathbf{x}) \mapsto \phi(t, \mathbf{x}, \mathbf{y})$ is everywhere nonpositive on the boundary of the cylinder $(0, \infty) \times \Omega_s$, the parabolic maximum principle then implies that $\phi(t, \mathbf{x}, \mathbf{y})$ is nonpositive in the whole cylinder $(0, \infty) \times \Omega_s \times \Omega_s$.

Claim : $\Delta \phi(t, \mathbf{x}, \mathbf{x}) \leq 0$ for all $(t, \mathbf{x}) \in (0, \infty) \times (\partial B)_s$.

For a sufficiently small $\delta > 0$, let $V = \{\psi(\mathbf{z}, \rho) := \mathbf{z} + \rho \nu(\mathbf{z}) ; \mathbf{z} \in \partial B \text{ and } 0 \leq \rho < \delta\}$ be the 1-sided δ -tubular neighborhood of ∂B . The Euclidean metric g_E can be expressed in V with respect to so-called Fermi coordinates $(\mathbf{z}, \rho) \in \partial B \times (0, \delta)$ as follows (see for instance [28, Lemma 3.1]) :

$$g_E = d\rho^2 + g_\rho,$$

where g_ρ is a Riemannian metric on the hypersurface $\Gamma_\rho = \{\mathbf{z} + \rho \nu(\mathbf{z}) ; \mathbf{z} \in \partial B\}$. Consequently, the Euclidean Laplacian in V takes on the following form with respect to Fermi coordinates :

$$\Delta = \frac{\partial^2}{\partial \rho^2} - H_\rho \frac{\partial}{\partial \rho} + \Delta_{g_\rho},$$

where H_ρ is the mean curvature of Γ_ρ and Δ_{g_ρ} is the Laplace-Beltrami operator of (Γ_ρ, g_ρ) .

Now, $K(t, \mathbf{x}, \mathbf{x}) = \sum_{k \geq 1} e^{-\lambda_k(\Omega)t} u_k(\mathbf{x})^2$, and it is known that for C^2 domains, $\|\nabla u_k\|_\infty$ is bounded by a constant times a finite power of λ_k (see [13, 16]). Hence, the functions $K(t, \mathbf{x}, \mathbf{x})$ and consequently $\phi(t, \mathbf{x}, \mathbf{x})$ vanish quadratically on $(\partial B)_s$. Thus, for any point $\mathbf{z} = \psi(\mathbf{z}, 0) \in (\partial B)_s$,

$$\frac{\partial}{\partial \rho} \phi(t, \psi(\mathbf{z}, \rho), \psi(\mathbf{z}, \rho)) \Big|_{\rho=0} = 0 \quad \text{and} \quad \Delta_{g_\rho} \phi(t, \psi(\mathbf{z}, \rho), \psi(\mathbf{z}, \rho)) \Big|_{\rho=0} = 0.$$

Therefore,

$$\Delta \phi(t, \mathbf{z}, \mathbf{z}) = \frac{\partial^2}{\partial \rho^2} \phi(t, \psi(\mathbf{z}, \rho), \psi(\mathbf{z}, \rho)) \Big|_{\rho=0},$$

which is nonpositive since $\frac{\partial}{\partial \rho} \phi(t, \psi(\mathbf{z}, \rho), \psi(\mathbf{z}, \rho)) \Big|_{\rho=0} = 0$ and, according to what we proved in the previous claim, the function $\rho \in [0, \delta) \mapsto \phi(t, \psi(\mathbf{z}, \rho), \psi(\mathbf{z}, \rho))$ achieves its maximum at $\rho = 0$.

Claim : Let τ be any positive real number. Except possibly for a finite set of values of t in $[\tau, \infty)$,

$$\frac{\partial}{\partial \varepsilon} Z_{\Omega_\varepsilon}(t) \Big|_{\varepsilon=0} > 0.$$

From the assumptions that B is convex and symmetric with respect to the hyperplane P , it follows that the product $V \cdot \nu(\mathbf{x})$ is positive at almost every point \mathbf{x} of $(\partial B)_s$. From equation (13) and the previous claim we then deduce that $\forall t > 0$,

$$\frac{\partial}{\partial \varepsilon} Z_{\Omega_\varepsilon}(t) \Big|_{\varepsilon=0} \geq 0.$$

To show that this quantity cannot vanish at more than a finite set of values of $t \in [\tau, \infty)$, we shall show that it is analytic as a function of t in the open right half plane, and positive for real values of t sufficiently large. By the unique continuation theorem an analytic function that vanishes on a set with a point of accumulation is identically zero, which would pose a contradiction.

To establish the analytic properties of $Z_{\Omega_\varepsilon}(t)$, we argue as follows. Observe first that the deformation Ω_ε depends analytically on ε in a

neighborhood of 0, since $\Omega_\varepsilon = f_\varepsilon(\Omega)$ with $f_\varepsilon(\mathbf{x}) = \mathbf{x} + \varepsilon X(\mathbf{x})$. As in the proof of Lemma 3.1 in [6], the Dirichlet Laplacian in Ω_ε is an analytic family of operators in the sense of Kato [20] with respect to the parameter ε . Because each eigenvalue of the Laplacian is at most finitely degenerate, according to [20, p. 425], there is a numbering of the eigenvalues $\{\lambda_k(\Omega_\varepsilon)\} \rightarrow \{\Lambda_k(\varepsilon)\}$ for which each $\Lambda_k(\varepsilon)$ is analytic in ε in a neighborhood of $\varepsilon = 0$. (Using this numbering, which is important only in a neighborhood of a degenerate eigenvalue, does not alter $Z(t)$ as defined in (1).) In consequence of the Hadamard formula for the derivative of an eigenvalue, $\frac{\partial \Lambda_k}{\partial \varepsilon}|_{\varepsilon=0}$ is dominated in norm by the integral of the square of the normal derivative of an associated L^2 normalized eigenfunction u_k over the boundary of the obstacle. We again call upon estimates for C^2 domains, by which both $\|u_k\|_\infty$ and $\|\nabla u_k\|_\infty$ are bounded by constants times finite powers of λ_k [13, 16], which in turn $\sim k^{\frac{2}{n}}$ by the Weyl law. It follows that both the series $\sum_{k \geq 1} e^{-\lambda_k(\Omega_\varepsilon)t}$ and its term-by-term derivative with respect to ε converge uniformly on each set of the form $\{\operatorname{Re} t \geq \tau > 0\}$, and are therefore analytic on such sets.

To finish the argument, we observe that by differentiating $Z_{\Omega_\varepsilon}(t) = \sum_{k \geq 1} e^{-\lambda_k(\Omega_\varepsilon)t}$,

$$\left. \frac{\partial}{\partial \varepsilon} Z_{\Omega_\varepsilon}(t) \right|_{\varepsilon=0} = e^{-\lambda_1(\Omega)t} \left(- \left. \frac{\partial \lambda_1(\Omega_\varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} + 0(e^{-(\lambda_2 - \lambda_1)t}) \right).$$

This is positive for large t because λ_1 is nondegenerate and $\left. \frac{\partial \lambda_1(\Omega_\varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} < 0$ by [15].

This completes the proof of (9). The proof of (10) and (11) relies on the formulae (7) and (8) that give for every $s \in \mathbb{R}^+$ and every $\varepsilon \neq 0$ sufficiently small,

$$\zeta_{\Omega_\varepsilon}(s) = R(s) + \frac{1}{\Gamma(s)} \int_0^1 \tilde{Z}_{\Omega_\varepsilon}(t) t^{s-1} dt + \frac{1}{\Gamma(s)} \int_1^\infty Z_{\Omega_\varepsilon}(t) t^{s-1} dt \quad (14)$$

and

$$\zeta'_{\Omega_\varepsilon}(0) = R'(0) + \int_0^1 \tilde{Z}_{\Omega_\varepsilon}(t) t^{-1} dt + \int_1^\infty Z_{\Omega_\varepsilon}(t) t^{-1} dt, \quad (15)$$

with $\det(\Omega_\varepsilon) = e^{-\zeta'_{\Omega_\varepsilon}(0)}$. □

Proof of Theorem 1.2. Let D be a bounded convex domain of \mathbb{R}^n and let $r > 0$ be less than the inradius of D . Observe first that for every $t > 0$, the function $\mathbf{x} \mapsto Z_{\Omega(\mathbf{x})}(t)$ is continuous on $\bar{D}_r = \{\mathbf{x} \in D :$

$\text{dist}(\mathbf{x}, \partial D) \geq r\}$. Indeed, we know that

$$\lambda_k(\Omega(\mathbf{x})) \geq \frac{n}{n+2} C_n \left(\frac{k}{|\Omega(\mathbf{x})|} \right)^{\frac{2}{n}}$$

(see [24]), where C_n is the constant appearing in Weyl's asymptotic formula and $|\Omega(\mathbf{x})|$ is the volume of $\Omega(\mathbf{x})$. Since $|\Omega(\mathbf{x})|$ does not depend on \mathbf{x} , we deduce that the series $\sum e^{-\lambda_k(\Omega(\mathbf{x}))t}$ converges uniformly on \bar{D}_r and that its sum $Z_{\Omega(\mathbf{x})}(t)$ depends continuously on \mathbf{x} . (The continuity of eigenvalues $\lambda_k(\Omega(\mathbf{x}))$ can be derived in several ways from standard continuity results cited in [17, Section 2.3.3]. In particular, see Remark 6.2 of [4].) Consequently, $Z_{\Omega(\mathbf{x})}(t)$ achieves its extremal values in \bar{D}_r .

Let $\mathbf{x} \in D_r = \{\mathbf{x} \in D : \text{dist}(\mathbf{x}, \partial D) > r\}$ be a point such that $\mathbf{x} \notin \heartsuit(D)$. From the definition of $\heartsuit(D)$, there exists a hyperplane of interior reflection P of D passing through \mathbf{x} . Moreover, since the reflection of $\partial D_s \setminus P$ is disjoint from ∂D_b , there exists $\delta > 0$ such that $\forall \varepsilon \in [0, \delta]$, the hyperplane P_ε parallel to P and passing through $\mathbf{x}_\varepsilon = \mathbf{x} - \varepsilon V$ is a hyperplane of interior reflection, where V is the unit vector perpendicular to P and pointing in the direction of D_s . Applying Proposition 2.1, we see that the function $\varepsilon \mapsto Z_{\Omega(\mathbf{x}_\varepsilon)}(t)$ is monotonically nonincreasing (notice that the variation formula (9) is given for a displacement into the small side D_s . Here, the obstacle moves in the opposite direction, that of $-V$, which has the effect of changing the sign of the derivative.) At the same time, the distance $\text{dist}(\mathbf{x}_\varepsilon, \heartsuit(D))$ is clearly decreasing since \mathbf{x}_ε moves into the big side. It follows that the set of points where $\mathbf{x} \mapsto Z_{\Omega(\mathbf{x})}(t)$ achieves its minimum cannot be disjoint from $\heartsuit(D)$. The minimum is therefore achieved at a point $\mathbf{x}_0(t) \in \heartsuit(D)$.

Similarly, if a point $\mathbf{x} \in D_r$ does not belong to the interior of $\heartsuit(D)$, then there exists a hyperplane of interior reflection passing through \mathbf{x} so that it is possible to move the obstacle into the small side D_s along a line segment perpendicular to P . The function $Z_{\Omega(\mathbf{x})}(t)$ is monotonically nondecreasing along such displacement while the obstacle approaches the boundary of D . Again, this proves that if the set of points where $\mathbf{x} \mapsto Z_{\Omega(\mathbf{x})}(t)$ achieves its maximum is not contained in the interior of $\heartsuit(D)$, then it must hit $\{\mathbf{x} \in D : \text{dist}(\mathbf{x}, \partial D) = r\}$.

The continuity of the zeta function and of the determinant in D_r derive from the continuity of the heat trace, through (7) and (8), according to which, for every $s \in \mathbb{R}^+$ and every $\mathbf{x} \in D_r$,

$$\zeta_{\Omega(\mathbf{x})}(s) = R(s) + \frac{1}{\Gamma(s)} \int_0^1 \tilde{Z}_{\Omega(\mathbf{x})}(t) t^{s-1} dt + \frac{1}{\Gamma(s)} \int_1^\infty Z_{\Omega(\mathbf{x})}(t) t^{s-1} dt, \quad (16)$$

and

$$\zeta'_{\Omega(\mathbf{x})}(0) = R'(0) + \int_0^1 \tilde{Z}_{\Omega(\mathbf{x})}(t) t^{-1} dt + \int_1^\infty Z_{\Omega(\mathbf{x})}(t) t^{-1} dt, \quad (17)$$

where $R(s)$ is a function that does not depend on \mathbf{x} . These formulae are not necessarily valid in the situation where the ball $B(\mathbf{x}, r)$ touches the boundary of D since, due to the cuspidal singularity that the domain $\Omega(\mathbf{x})$ will then present, in consequence of which the function $\tilde{Z}_{\Omega(\mathbf{x})}(t)/\sqrt{t}$ may cease to be bounded in the neighborhood of $t = 0$.

Let $\mathbf{x} \in D_r$ be a point lying outside $\heartsuit(D)$ and let $s \in \mathbb{R}^+$. As before, using the definition of $\heartsuit(D)$ and Proposition 2.1, we see that it is possible to move $B(\mathbf{x}, r)$ towards the heart so as to decrease $\zeta_{\Omega(\mathbf{x})}(s)$. This enables us to construct a (possibly finite) sequence of points converging to a point $\mathbf{y} \in \heartsuit(D)$, along which the zeta function is decreasing. Thus, $\zeta_{\Omega(\mathbf{x})}(s) < \zeta_{\Omega(\mathbf{y})}(s)$, which leads to (3). Similarly, it is possible to move $B(\mathbf{x}, r)$ in the direction of the boundary so as to increase $\zeta_{\Omega(\mathbf{x})}(s)$, which implies that $\zeta_{\Omega(\mathbf{x})}(s)$ is less than a limiting value of $\zeta_{\Omega(\mathbf{y})}(s)$ as $B(\mathbf{y}, r)$ approaches the boundary of D .

Now, when $r < \text{dist}(\heartsuit(D), \partial D)$, the heart is contained in D_r and, consequently, $\mathbf{x} \mapsto \zeta_{\Omega(\mathbf{x})}(s)$ is continuous on $\heartsuit(D)$ (which is a compact set) and achieves its minimum at a point $\mathbf{x}_0(s) \in \heartsuit(D)$. Similarly, $\mathbf{x} \in \heartsuit(D) \mapsto \zeta_{\Omega(\mathbf{x})}(s)$ achieves its maximum at a point which belongs to the interior of $\heartsuit(D)$, since a ball of radius $r < \text{dist}(\heartsuit(D), \partial D)$ centered at the boundary of $\heartsuit(D)$ can be moved away from $\heartsuit(D)$ so as to increase ζ .

The statement concerning the determinant can be proved using the same arguments.

□

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